Journal of Engineering Physics and Thermophysics, Vol. 82, No. 3, 2009

SIMPLIFICATION OF THE MATHEMATICAL MODELS OF OBJECTS IN SIMULATOR COMPLEXES

Men′ Tsinsun and K. M. Shestakov

UDC 681.5:519.711.3

Mathematical models of control systems are important for problems that are solved in real time in the software for simulator complexes of complicated machinery (pilot simulators, transport simulators, etc.). Simplification of the models by excluding weakly affecting components allows one to reduce loading on the computational medium of simulators. The Schur real decomposition approximates a simplified model to the model of the initial system. A new method to increase the quality of approximation with allowance for transient and established values of object reaction, that is, approximation of transfer functions, is suggested. It is made by decreasing deviations over the matrices of a transfer function between the initial and simplified models. The applicability and effectiveness of this method of simplification are proved using a dynamic model of a pilot system as an example.

Keywords: *simplification of a model, Schur real decomposition, controllability gramian, observability gramian.*

Introduction. In constructing simulator complexes, the problem of adaptation of the mathematical models of dynamic objects and of the processes occurring in control systems in real time is being solved. Usually, the time amounts to less than 10 msec. The number of such initiated processes ranges up to several tens. Such a level can be attained only through simplification of mathematical models. Among the ways of obtaining the desired result is the exclusion of weakly affecting components of equations without appreciable losses in the behavior of objects.

At the present time, there exist various methods for simplification of models, but some of them are unable to ensure the stability of systems at a given accuracy, and others could not be applied in multi-dimensional systems. In [1], for an asymptotically stable [2] system of the matrices of coefficients $\{A, B, C\}$, after the Schur decomposition the matrix A is transformed into an upper triangular form. Its diagonal elements are eigenvalues of the matrix, and they are arranged in decreasing order in real components [3]. We will designate the system obtained as $\{A_s, B_s, C_s\}$, where

$$A_s = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B_s = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C_s = \begin{bmatrix} C_1 & C_2 \end{bmatrix} .$$

If $\mathrm{Re}\{\lambda_i(A_{11})\} >> \mathrm{Re}\{\lambda_j(A_{22})\}$ ($\mathrm{Re}\{\lambda(\cdot)\}$ are the real parts of the eigenvalues of the matrix), the simplified model is defined as $\{A_{11}, B_1 - A_{12}A_{22}^{-1}B_2, CA_s^{-1}B_s[A_{11}^{-1}(B_1 - A_{12}A_{22}^{-1}B_2)]^+\}$. Sometimes the computation of the inverse matrix leads to a large number of conditionality [4] and to difficulty in realizing the space of the states of the system, if the matrix A has complex eigenvalues. This fact puts the given method into the group of "unstable" methods. Since the orthogonal operator does not impair the conditionality of matrices, below a more "stable" method of simplification of systems will be considered.

1. Algorithms and Error of Simplification. In addition to stable poles, the transfer function of a practical system often contains chains of integrating type (i.e., $s = 0$). For linear stationary and multidimensional systems with an integrating-type chain it is necessary to preserve all the poles so that the simplified system could be as close to the initial one as possible. Since controllability and observability gramians are nonexistent for such a system, in what follows we will consider systems of two types encountered in practice. For the convenience of readers, below some of the terms used in describing control systems are explained.

Controllability [5]: the system is called fully controllable if it can be transferred from any initial state $X(0)$ into the coordinate origin (0, 0, ..., 0) under control $U(t)$ in a finite time.

Belarusian State University, 8 Kurchatov Str., Minsk, 220108, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 82, No. 3, pp. 590–599, May–June, 2009. Original article submitted June 11, 2008; revision submitted October 16, 1008.

1062-0125/09/8203-0588©2009 Springer Science+Business Media, Inc.

Observability [5]: the system is called fully observable if from the results of measurement of the input $U(t)$ and output $Y(t)$ variables it is possible to unambiguously determine all the components of the vector $X(t)$ in the finite interval of time.

Gramian [2]: this is a matrix whose elements are paired scalar products of a system of vectors checked up on linear independence. Such a matrix is also called the Gram matrix for a system of vectors.

Controllability gramian [2]: this is an extended interpretation of the Gram matrix in application to a system of functions that are the states of a linear dynamic system. The controllability gramian can be found from the following Lyapunov equation:

$$P \cdot A^T + A \cdot P + B \cdot B^T = 0 .$$

Observability gramian [2]: according to the Kalman duality principle, a dual system is introduced into consideration, the variable states of which form its own Gram matrix under the conditions analogous to those considered above. The observability gramian can be found from the following Lyapunov equation:

$$Q \cdot A + A^T \cdot Q + C^T \cdot C = 0 .$$

1.1. System without integrating-type chains. We consider a linear stationary system which is asymptotically stable, quite controllable, and observable:

$$S_1 : \begin{cases} \dot{X} = AX + BU , \\ Y = CX , \end{cases} \tag{1}$$

where $X \in R^{n \times 1}$, $U \in R^{m \times 1}$, $Y \in R^{p \times 1}$ are the vectors of phase coordinates, of controlling and regulated quantities; $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$ are the matrices of constant coefficients.

We will designate the eigenvalues of the matrix A located in decreasing order in real parts in the form of λ_1, λ_2, ..., λ_n and

$$\mathrm{Re}\left\{\lambda_k\right\} >> \mathrm{Re}\left\{\lambda_{k+1}\right\} , \tag{2}$$

where $k < n$.

The Schur real decomposition can in essence be reduced to the process of Gram–Schmidt orthogonalization, whereas the real variant of the Schur decomposition theorem is described as follows: for any matrix $A \in R^{n \times n}$ there exists a real orthogonal matrix $U \in R^{n \times n}$ such that

$$U^T A U = \begin{bmatrix} A_1 & * & symbol & * \\ 0 & A_2 & symbol & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & symbol & A_k \end{bmatrix},$$

where for each i the matrix A_i (i = 1, 2, ..., k) has the size 1 × 1 or 2 × 2 correspondingly conforming to the real eigenvalue or unreal pair of complex-conjugate eigenvalues of the matrix A. The blocks A_i can be arranged in any prescribed order [3].

Thus, according to the Schur theorem of real decomposition there exits the real orthogonal matrix U that satisfies the following condition:

$$U^T A U = S = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{3}$$

where $A_{11} \in R^{k \times k}$; $A_{22} \in R^{(n-k)\times(n-k)}$; $\lambda_i(A_{11}) = \lambda_i(A)$, i = 1, 2, ..., k; $U^T = U^{-1} \in R^{n \times n}$, whereas $\lambda(A_{11})$ and $\lambda(A)$ are the eigenvalues of the matrices A_{11} and A.

Let us perform similarity transformation for the matrix S to transform it into a block diagonal one, diag(A_{11}, A_{22}). In so doing, we introduce the transforming matrix V as

$$V = \begin{bmatrix} I_k & \tilde{X} \\ 0 & I_{n-k} \end{bmatrix}_{n\times n}, \tag{4}$$

where $\tilde{X} \in R^{k\times(n-k)}$. The inverse matrix V^{-1} will be defined in the form

$$V^{-1} = \begin{bmatrix} I_k & -\tilde{X} \\ 0 & I_{n-k} \end{bmatrix}_{n\times n}, \tag{5}$$

i.e., the relation $VV^{-1} = V^{-1}V = I_n$ has been satisfied.

In order that

$$V^{-1}SV = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \tag{6}$$

it is necessary to satisfy the Sylvester equation [4]:

$$A_{11}\tilde{X} + A_{12} - \tilde{X}A_{22} = 0\,. \tag{7}$$

Equation (7) has been obtained by substitution of Eqs. (3)–(5) into Eq. (6).

In the case of nonsingular transformation $Z = T^{-1}X$ the initial system S_1 will take the form

$$S_2: \begin{cases} \dot{Z} = A_t Z + B_t U\,, \\ Y = C_t Z\,, \end{cases} \tag{8}$$

where $T = UV$; $A_t = T^{-1}AT \in R^{n\times n}$; $B_n = T^{-1}B \in R^{n\times m}$; $C_t = CT \in R^{p\times n}$; $Z \in R^{n\times 1}$; A_t, B_t, and C_t are the matrices of constant coefficients.

We will denote

$$A_t = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad B_t = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C_t = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \tag{9}$$

where $A_{11} \in R^{k\times k}$; $A_{22} \in R^{(n-k)\times(n-k)}$; $B_1 \in R^{k\times m}$; $B_2 \in R^{(n-k)\times m}$; $C_1 \in R^{p\times k}$; $C_2 \in R^{p\times(n-k)}$; $Z_1 \in R^{k\times 1}$; $Z_2 \in R^{(n-k)\times 1}$.

Since the controllability and observability properties do not depend on the choice of the coordinate system [5], the system S_2 is asymptotically stable, quite controllable, and observable. Because the matrix U is orthogonal and the solution of Eq. (7) does not degrade the conditionality [4], the transforming matrix $T = UV$ is also satisfactorily conditioned.

We introduce the notation

$$G(s) = C(sI_n - A)^{-1}B = C_t(sI_n - A_t)^{-1}B_t\,, \quad G_1(s) = C_1(sI_k - A_{11})^{-1}B_1\,,$$

$$G_2(s) = C_2(sI_{n-k} - A_{22})^{-1}B_2\,. \tag{10}$$

As a result we have

$$G(s) = G_1(s) + G_2(s)\,, \tag{11}$$

i.e., the initial system S_1 is divided into two independent subsystems ($G_1(s)$ and $G_2(s)$), so that the simplified system S_3 containing the principal poles has been obtained in the following form:

$$S_3 : \begin{cases} \dot{Z}_1 = A_{11} Z_1 + B_1 U , \\ \hat{Y} = C_1 Z_1 , \end{cases} \tag{12}$$

where $\hat{Y} \in R^{p\times 1}$, and the requirement that $\hat{Y} = Y$ has been satisfied.

As a result, we come to the conclusion: the simplified system S_3 is asymptotically stable, quite controllable, and observable; it also contains the principal poles of the initial system S_1.

If the definition of the Hankel singular numbers is given for the models S_2 and $G(s)$ in the form

$$\sigma_i \, [G(s)] = \left\{ \lambda \, (P_t Q_t) \right\}^{1/2} , \quad i = 1, 2, ..., n ,$$

and conditionally $\sigma_i(\cdot) \geq \sigma_{i+1}(\cdot)$, where P_t and Q_t are the controllability and observability gramians of the system S_2, then the upper limit of the error of the given algorithm is determined [6] in the form

$$\| G(j\omega) - G_1(j\omega) \|_{L^\infty} = \| G_2(j\omega) \|_{L^\infty} \leq 2 \sum_{i=1}^{n-k} \lambda_i^{1/2} (P_{22} Q_{22})$$

and

$$\sum_{i=k+1}^{n} \sigma_i \, [G(s)] \leq \sum_{i=1}^{n-k} \sigma_i \, [G_2(s)] \leq \sum_{i=1}^{n-k} \sigma_i \, [G(s)] \, .$$

1.2. System with integrating-type chains. We consider the system $S_1(1)$ on the assumption that the matrix A has m zero eigenvalues $(1 \leq m \leq n)$ and $(n - m)$ eigenvalues with negative real parts. The idea to simplify the system with integrating-type chains is as follows: first we divide the initial system into two independent subsystems $\{A_0, B_0, C_0\}$ and $\{A_1, B_1, C_1\}$, where A_0 is the zero matrix and A_1 is the matrix, the real parts of all the eigenvalues for which are negative; thereafter we apply the above-indicated method of simplification to the subsystem $\{A_1, B_1, C_1\}$ and obtain the simplified subsystem $\{A_r, B_r, C_r\}$; the final simplified system will have the form $\{A_r, B_r, C_r\} + \{A_0, B_0, C_0\}$.

1. We will divide the initial system into two subsystems using two methods:

I) the method of singular value decomposition (SVD) [3] of the matrix A.

Let

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T ,$$

where $U^T = U^{-1} \in R^{n\times n}$; $V^T = V^{-1} \in R^{n\times n}$; $\Sigma \in R^{(n-m)\times(n-m)}$ is the matrix with positive diagonal elements. Thus,

$$V^T A V = V^T U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T V = \begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}_{n\times n} ,$$

where $A_1 \in R^{(n-m)\times(n-m)}$; $A_2 \in R^{m\times(n-m)}$.

If $A_2 \neq 0$, then let the matrix P_s have the form

$$P_s = \begin{bmatrix} I_{n-m} & 0 \\ A_2 A_1^{-1} & I_m \end{bmatrix} , \tag{13}$$

and its inverse matrix

$$P_s^{-1} = \begin{bmatrix} I_{n-m} & 0 \\ -A_2 A_1^{-1} & I_m \end{bmatrix} . \tag{14}$$

As a result we obtain

$$P_s^{-1}V^T AVP_s = P_s^{-1}\begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix} P_s = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $A_2 = 0$, then we take $P_s = I_n$ and denote $T_f = VP_s$. In this case

$$A_f = T_f^{-1} AT_f = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_0 \end{bmatrix}, \tag{15}$$

$$B_f = T_f^{-1} B = \begin{bmatrix} B_1 \\ B_0 \end{bmatrix}, \quad C_f = CT_f = \begin{bmatrix} C_1 & C_0 \end{bmatrix}, \tag{16}$$

where $A_0 = 0_m$; $B_1 \in R^{(n-m)\times m}$; $B_0 \in R^{m\times m}$; $C_1 \in R^{p\times(n-m)}$; $C_0 \in R^{p\times m}$.

II) The method of orthonormalized basis for the null space [3] of the matrix.

Let Q_1 be the orthonormalized basis for the null space of the matrix A, Q_2 be the orthonormalized basis for the null space of the matrix Q_1^T, i.e., $AQ_1 = 0$, $Q_1^T Q_1 = I_m$, $Q_1^T Q_2 = 0$, $Q_2^T Q_2 = I_{n-m}$, where $Q_1 \in R^{n\times m}$, $Q_2 \in R^{n\times(n-m)}$.

Since $Q_s = [Q_2 Q_1]$ is the orthogonal matrix, we obtain

$$Q_s^T AQ_s = \begin{bmatrix} Q_2^T AQ_2 & 0 \\ Q_1^T AQ_2 & 0 \end{bmatrix}_{n\times n}.$$

Let $A_1 = Q_2^T AQ_2$ and $A_2 = Q_1^T AQ_2$. Similarly to the above, the initial system S_1 can also be transformed to the forms of (15) and (16). With the aid of the Matlab program one can easily carry out singular decomposition of the matrix A or obtain the matrices Q_1 and Q_2.

2. We will simplify the system $\{A_1, B_1, C_1\}$ by means of the Schur decomposition.

If we denote the Hankel singular numbers of the system $\{A_1, B_1, C_1\}$ by σ_i ($i = 1, 2, \ldots, n-m$) and satisfy the condition

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > \sigma_{k+1} \geq \ldots \geq \sigma_{n-m} > 0,$$

we will be able to obtain the k-dimensional ($k < n-m$) simplified model $\{A_r, B_r, C_r\}$ by means of the Schur real decomposition.

If we denote

$$G(s) = C(sI_n - A)^{-1} B, \quad G_0(s) = C_0(sI_m - A_0)^{-1} B_0, \quad G_1(s) = C_1(sI_{n-m} - A_1)^{-1} B_1,$$

$$G_r(s) = C_r(sI_k - A_r)^{-1} B_r,$$

then the reduced system will take the form

$$G_R(s) = G_r(s) + G_0(s);$$

the realization of the space of states can be represented as

$$\left\{ \begin{bmatrix} A_r & 0 \\ 0 & A_0 \end{bmatrix}, \begin{bmatrix} B_r \\ B_0 \end{bmatrix}, \begin{bmatrix} C_r & C_0 \end{bmatrix} \right\},$$

and the estimation of the error of simplification is

$$\overline{\sigma}\,[G\,(j\omega) - G_R\,(j\omega)] \le 2 \sum_{i=k+1}^{n-m} \sigma_i\,, \quad \forall\omega\,.$$

2. Approximation of the Transfer Functions between the Models of the Simplified and Initial Systems. If in the dynamic characteristics of the system there are components that correspond to small eigenvalues λ_i ($i = k+1,\ k+2,\ ...,\ n$), we will consider that $\dot{Z}_2 = 0$. Then from Eqs. (8) and (9) we obtain

$$\dot{Z}_1 = A_{11}Z_1 + B_1U\,,\quad 0 = A_{22}Z_2 + B_2U\,,\quad Y = C_1Z_1 + C_2Z_2\,. \tag{17}$$

Further we will consider two cases:

1) if the matrix A_{22} is reversible, then after the replacement of Z_2 Eq. (17) will have the form

$$S_4 : \begin{cases} \dot{Z}_1 = A_RZ_1 + B_RU\,, \\ Y = C_RZ_1 + D_RU\,, \end{cases} \tag{18}$$

where $Z_1 \in R^{k\times 1}$; A_R, R_R, C_R, and D_R are the constant matrices; $A_R = A_{11}$; $B_R = B_1$; $C_R = C_1$; $D_R = -C_2A_{22}^{-1}B_2$;

2) if A_{22} is irreversible, we will replace the matrix A_{22}^{-1} by the generalized inverse one A_{22}^{+}.

It can be easily noted that the simplified model S_4 differs from the initial system S_1, and since the matrix $(-C_2A_{22}^{-1}B_2)$ is not equal in principle to the zero matrix in the initial system, then the initial values of the output quantities of the system in the model S_4 are not equal to the zero values of the initial system.

In order to obtain equal initial values and approximate established values for the simplified model S_4, as compared to the model of the initial system S_1, it is necessary to introduce correction to the form of the simplified model S_4 in the following way. First, we will write the simplified model S_4 in the form

$$S_5 : \begin{cases} \dot{Z}_1 = A_RZ_1 + B'_RU\,, \\ Y = C_RZ_1\,, \end{cases} \tag{19}$$

where $B'_R \in R^{k\times m}$, then we introduce approximation of transfer functions between the models of the simplified and initial systems.

Using the idea of the Padé approximation, we decompose the matrix of the transfer function for the initial system in the form

$$G\,(s) = C\,(sI_n - A)^{-1}\,B = -\,CA^{-1}\,B - CA^{-2}\,Bs - CA^{-3}\,Bs^2 - \ldots\,.$$

Similarly we decompose the matrix of the transfer function for the simplified system in the form

$$G_R\,(s) = C_R\,(sI_k - A_R)^{-1}\,B'_R = -\,C_RA_R^{-1}\,B'_R - C_RA_R^{-2}\,B'_Rs - C_RA_R^{-3}\,B'_Rs^2 - \ldots\,.$$

Approximate $G_R(s)$ to $G(s)$ and determine the matrix B'_R on coordination of the corresponding terms.

It is necessary to satisfy the following condition to ensure equal or more approximate established values for the two models (S_1 and S_5):

$$\begin{bmatrix} CA^{-1}B \\ CA^{-2}B \\ \vdots \\ CA^{-j}B \end{bmatrix} = \begin{bmatrix} C_RA_R^{-1} \\ C_RA_R^{-2} \\ \vdots \\ C_RA_R^{-j} \end{bmatrix} B'_R\,,\quad j = 1, 2, \ldots\,, \tag{20}$$

where $G_R(s) = G_R(sI_k - A_R)^{-1}B'_R = -\sum_{i=0}^{\infty} C_R A_R^{-i-1} B'_R s^i$; $G(s) = C(sI_n - A)^{-1}B = -\sum_{i=0}^{\infty} CA^{-i-1}Bs^i$ are the matrices of the transfer function for the models S_5 and S_1.

Let

$$B'_R = \begin{bmatrix} b'_{11} & b'_{12} & symbol & b'_{1m} \\ b'_{21} & b'_{22} & symbol & b'_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b'_{k1} & b'_{k2} & symbol & b'_{km} \end{bmatrix}_{k\times m} .$$

This means that in the matrix equation (20) there are $k \times m$ unknown values (b'_{ij}, $i = 1, 2, ..., k$, $j = 1, 2, ..., m$) and $(j \times p) \times m$ scalar equations. We must solve the matrix equation (20) to obtain the matrix B'_R and the final simplified model S_5.

We will consider the problem of solving the matrix equation (20) in the following two cases:

a) if $k/p = j$ and j is equal to the integer number, then the matrix B'_R is determined singly after cutting off of the first j lines in the matrix equation (20);

b) if $k/p = j$ and j is not equal to an integer number, then after the cutting off of the first k lines we will obtain close solutions of the matrix equation (20). Incidentally, for solving the matrix equation (20) one can also use the least squares method.

3. Sequence of Programing to Simplify a Model. Proceeding from the indicated method, the reduction will be made on taking the following steps:

1) determine the asymptotic stability, full controllability, and full observability of the system S_1, but if all the conditions are complied with, we pass to the next step;

2) using Eq. (3), transform the matrix A of the initial system into the upper triangular matrix S with the aid of the Schur real decomposition;

3) from Eqs. (8) and (9) obtain the model of system S_2 after solving Eq. (7);

4) from Eq. (12) presented after the division of the states of the system into two independent subsystems, obtain the model of system S_3;

5) from Eq. (18) obtain the model of system S_4;

6) from the distribution of all the eigenvalues of the system determine the order of the simplified model S_5;

7) solve the matrix equation (20) and determine the unknown matrix B'_R;

8) from Eq. (19) obtain the final simplified model S_5;

9) introduce mathematical simulation of an example with the aid of the Matlab program.

4. Example of Imitation. The dynamic model of the linearized pilot system by the side motion is known in the following matrix form:

$$\dot{x}_1 = -0.422x_1 + 0.068x_2 + 0.037x_3 - x_4 + 0.0003x_5 + 0.0005x_6 ,$$

$$\dot{x}_2 = x_3 + 0.004x_4 ,$$

$$\dot{x}_3 = -31x_1 - 3.8123x_3 + 0.5646x_4 - 0.6333x_5 + 0.1429x_6 ,$$

$$\dot{x}_4 = 7.5347x_1 - 0.024x_3 - 0.4735x_4 - 0.0314x_5 - 0.0618x_6 ,$$

$$\dot{x}_5 = -19.2x_5 + 19.2u_1 ,$$

$$\dot{x}_6 = -19.2x_6 + 19.2u_2 ,$$

$$y_1 = 57.2958 x_1 ,$$

$$y_2 = 57.2958 x_2 ,$$

where x_1, x_2, x_3, x_4, x_5, and x_6 are the phase coordinates of the system; u_1 and u_2 are the controlling quantities; y_1 and y_2 are the regulated quantities; x_1 is the angle of deviation in the course; x_2 is the angle of turning in roll; x_3, angular velocity in turning in roll; x_4, angular velocity in the course; x_5, angle of deviation of the aileron; x_6, angle of deviation of the rudder; u_1, input value of the tracking of aileron; u_2, input value of the tracking of rudder; y_1, angle of deviation in the course; y_2, angle of turning in roll.

With the aid of the Matlab program we obtain all eigenvalues of the matrix A for the systems S_1 and the real upper triangular matrix S after the Schur decomposition in the following form:

$$\lambda = -0.0216 , \quad -0.4720 + 2.9073i , \quad -0.4720 - 2.9073i , \quad -3.7422 , \quad -19.2000 , \quad -19.2000 ;$$

$$S = \begin{bmatrix} -0.0216 & -0.9297 & 0.1842 & -1.2771 & -0.0117 & 0.0072 \\ 0 & -0.3057 & 9.2662 & -30.4884 & -0.6068 & 0.1520 \\ 0 & -0.9152 & -0.6383 & 1.3255 & -0.1448 & -0.0321 \\ 0 & 0 & 0 & -3.7422 & -0.1128 & 0.0079 \\ 0 & 0 & 0 & 0 & -19.2000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -19.2000 \end{bmatrix} .$$

From the distribution of these eigenvalues we will select the order of the simplified system S_5 equal to 4 (i.e., $k = 4$); at the same time, the first four eigenvalues by absolute value of the real parts comprise $\sum_{i=1}^{4} |\sigma_i| / \sum_{j=1}^{6} |\sigma_j| \approx 10.92\%$.

After the transformation of the matrix A into the upper triangular matrix S with the aid of the Schur real decomposition and solution of Eq. (7) we obtain the model S_3 from Eq. (12):

$$\dot{x}_1 = -0.0216 x_1 - 0.9297 x_2 + 0.1842 x_3 - 1.2771 x_4 - 0.0560 u_1 + 0.0169 u_2 ,$$

$$\dot{x}_2 = -0.3057 x_2 + 9.2662 x_3 - 30.4884 x_4 - 0.7559 u_1 + 0.1825 u_2 ,$$

$$\dot{x}_3 = -0.9152 x_2 - 0.6383 x_3 + 1.3255 x_4 - 0.1770 u_1 - 0.0249 u_2 ,$$

$$\dot{x}_4 = -3.7422 x_4 - 0.1401 u_1 + 0.0098 u_2 ,$$

$$y_1 = -0.2211 x_1 - 6.1639 x_2 - 16.6925 x_3 + 54.4622 x_4 ,$$

$$y_2 = -57.1587 x_1 + 2.0622 x_2 - 3.2334 x_3 - 0.9896 x_4 ,$$

as well as the model S_4 from Eq. (18):

$$\dot{x}_1 = -0.0216 x_1 - 0.9297 x_2 + 0.1842 x_3 - 1.2771 x_4 - 0.0560 u_1 + 0.0169 u_2 ,$$

$$\dot{x}_2 = -0.3057 x_2 + 9.2662 x_3 - 30.4884 x_4 - 0.7559 u_1 + 0.1825 u_2 ,$$

$$\dot{x}_3 = -0.9152 x_2 - 0.6383 x_3 + 1.3255 x_4 - 0.1770 u_1 - 0.0249 u_2 ,$$

$$\dot{x}_4 = -3.7422 x_4 - 0.1401 u_1 + 0.0098 u_2 ,$$

$$y_1 = -0.2211 x_1 - 6.1639 x_2 - 16.6925 x_3 + 54.4622 x_4 + 0.0002 u_1 + 0.0092 u_2 ,$$

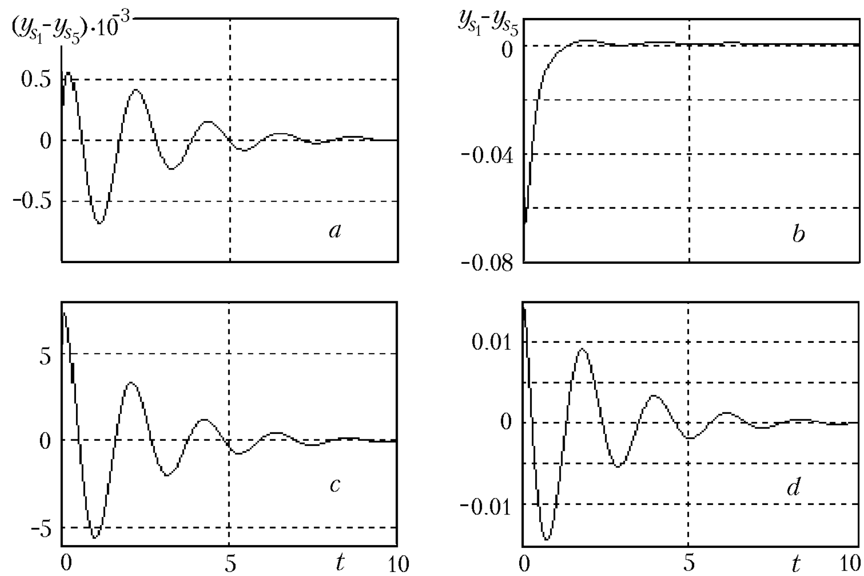

Fig. 1. Graphs of differences of transition functions between the models of systems S_1 and S_5 (k = 4): a) y_1 of u_1; b) y_2 of u_1; c) y_1 of u_2; d) y_2 of u_2. t, sec.

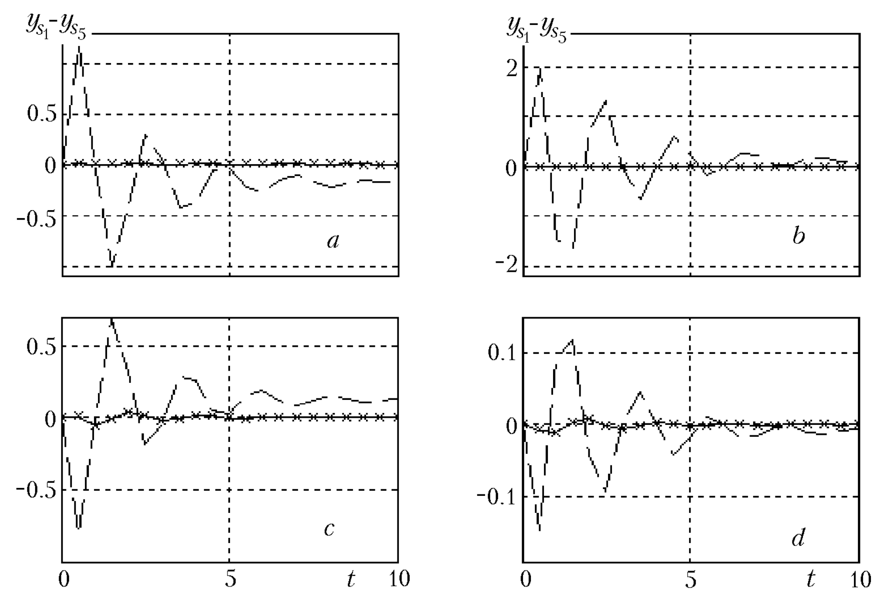

Fig. 2. Graphs of differences of transition functions between the models of systems S_1 and S_5 at different orders of system S_5 (k = 3, 4, 5): a) y_1 of u_1; b) y_2 of u_1; c) y_1 of u_2; 2) y_2 of u_2; k = 3, 4, 5 correspond to the dashed and continuous lines and the line with crosses. t, sec.

$$y_2 = -57.1587x_1 + 2.0622x_2 - 3.2334x_3 - 0.9896x_4 - 0.1227u_1 + 0.0271u_2 .$$

After solution of the matrix equation (20), the final simplified model S_5 is determined from Eq. (19):

$$\dot{x}_1 = -0.0216x_1 - 0.9297x_2 + 0.1842x_3 - 1.2771x_4 - 0.0473u_1 + 0.0153u_2 ,$$

$$\dot{x}_2 = -0.3057x_2 + 9.2662x_3 - 30.4884x_4 - 0.7262u_1 + 0.1817u_2 ,$$

$$\dot{x}_3 = -0.9152x_2 - 0.6383x_3 + 1.3255x_4 - 0.1707u_1 - 0.0264u_2 ,$$

$$\dot{x}_4 = -3.7422x_4 - 0.1348u_1 + 0.0094u_2\,,$$

$$y_1 = -0.2211x_1 - 6.1639x_2 - 16.6925x_3 + 54.4622x_4\,,$$

$$y_2 = -57.1587x_1 + 2.0622x_2 - 3.2334x_3 - 0.9896x_4\,.$$

With the aid of the Matlab program we obtained graphs of the differences of transition functions between the models of systems S_1 and S_5 (k = 4, see Fig. 1). It is seen that the model S_5 approximates well the initial system S_1 after application of the above-indicated method.

We will also present graphs of the differences of transition functions between the models of systems S_1 and S_5 under the conditions of different orders of the system S_5 (i.e., k = 3, 4, 5) (see Fig. 2). It is seen that the differences of the transition functions between the models of systems S_1 and S_5 (i.e., k = 3) depend on the order k of the simplified model S_5: the smaller the order k, the greater the difference, but the lower the efficiency of simplification.

Conclusions. By the method of approximation of transfer functions the model of linear and multidimensional system has been simplified. The proposed method of simplification of mathematical models of systems allows one to raise the accuracy of approximation of a simplified model to the initial one over transient and established values, to decrease the time of calculation of the system parameters in real regime, and to ease designing of controlling systems in theoretical synthesis. The above-indicated theoretical analysis and simulation of an example prove the applicability and efficiency of this method of simplification.

NOTATION

A_0, zero matrix; A, B, C, A_t, B_t, C_t, A_R, B_R, C_R, D_R, A_f, B_f, and C_f, matrices of constant coefficients; $\{A, B, C\}$, $\{A_1, B_1, C_1\}$, $\{A_r, B_r, C_r\}$, $\{A_0, B_0, C_0\}$, and $\{A_t, B_t, C_t\}$, designations of systems in a matrix form; $G(s)$, $G_1(s)$, $G_2(s)$, $G_0(s)$, $G_r(s)$, and $G_R(s)$, matrices of transfer function; i, imaginary unit; k, order of simplified model S_5; n, m, p, orders of columns of phase coordinates of controlling and regulated quantities; P_t, Q_t, controllability and observability gramians of system $\{A_t, B_t, C_t\}$ obtainable after the solution of the Lyapunov matrix equations $A_tP_t + P_tA_t^T + B_tB_t^T = 0$, $A_t^TQ_t + Q_tA_t + C_t^TC_t = 0$; Q_1, Q_2, orthonormalized bases of the right zero space; Q_s, orthogonal matrix; $\mathrm{Re}\{\lambda(\cdot)\}$, real parts of eigenvalues of a matrix; S, V, K, P_s, B_R', matrices; u_1, u_2, controlling quantities of the system; T, transposition of a matrix; t, time; x_1, x_2, x_3, x_4, x_5, and x_6, phase coordinates of the system; X, U, Y, vectors of phase coordinates of controlling and regulated quantities for a control system; y_1, y_2, regulated quantities of a system; Z, vector; λ, all poles of initial system; λ_1, λ_2, ..., λ_n, eigenvalues of a matrix; $\sigma_i[G(s)]$, Hankel singular numbers for $G(s)$; Σ, matrix with diagonal positive elements.

REFERENCES

1. A. Rachid and G. Hashim, Model reduction via Schur decomposition, *IEEE Trans. Autom. Contr.*, **37**, No. 5, 666–668 (1992).
2. Yu. N. Andreev, *Management of Finite-Dimensional Linear Objects* [in Russian], Nauka, Moscow (1976).
3. R. Horn and Ch. Johnson, *Matrix Analysis* [Russian translation], Mir, Moscow (1989).
4. G. Golub and Ch. Van Loan, *Matrix Computations* [Russian translation], Mir, Moscow (1999), pp. 343–352.
5. A. A. Voronov, D. P. Kim, V. M. Lokhin, et al., *Theory of Automatic Control* [in Russian], Pt. II, Vysshaya Shkola, Moscow (1986), pp. 304–318.
6. K. Glover, All optimal Hankel-norm approximation of linear multivariable systems and their L^∞-error bounds, *Int. J. Control*, **39**, No. 6, 1115–1193 (1984).